

Critical velocity in cylindrical Bose-Einstein condensates

P.O. Fedichev^{1,3} and G.V. Shlyapnikov^{2,3}

¹*Institut f. Theoretische Physik, Universität Innsbruck, Technikerstr. 25/2, A6020, Innsbruck, Austria*

²*FOM Institute for Atomic and Molecular Physics, Kruislaan 407, 1098SJ Amsterdam, The Netherlands*

³*Russian Research Center Kurchatov Institute, Kurchatov Square, 123182 Moscow, Russia*

We describe a dramatic decrease of the critical velocity in elongated cylindrical Bose-Einstein condensates which originates from the non-uniform character of the radial density profile. We discuss this mechanism with respect to recent measurements at MIT.

Superfluidity is one of the striking manifestations of quantum statistics, and the occurrence of this phenomenon depends on the excitation spectrum of the quantum liquid. The key physical quantity is the critical velocity v_c , that is, the maximum velocity at which the flow of the liquid is still non-dissipative (superfluid). The well-known Landau criterion [1] gives the critical velocity as the minimum ratio of energy to momentum in the excitation spectrum:

$$v_c = \min \left(\frac{\epsilon(k)}{k} \right) \quad (1)$$

(we put $\hbar = 1$ and the particle mass $M = 1$). Actually, in liquid ^4He the Landau criterion overestimates the critical velocity. The explanation of this fact was put forward by Feynman (see e.g. [2]) who suggested that superfluidity is destroyed by spontaneous creation of complex excitations (vortex lines, vortex rings, etc.). Extensive theoretical and experimental studies on this subject are reviewed in [3].

Bose-Einstein condensation of dilute trapped clouds of alkali atoms [4–6] offers new possibilities for the investigation of superfluidity [7]. In the spatially homogeneous case, the spectrum of elementary excitations of a Bose-condensed gas is given by the Bogolyubov dispersion law [8]:

$$\epsilon(k) = \sqrt{\left(\frac{k^2}{2}\right)^2 + 2c_s^2 \frac{k^2}{2}}, \quad (2)$$

and the Landau critical velocity is equal to the speed of sound $c_s = \sqrt{gn_0}$ (n_0 is the condensate density, $g = 4\pi a$, and $a > 0$ is the scattering length). The first experimental observation of the critical velocity in trapped gaseous condensates, recently reported by the MIT group [9], gives a significantly smaller value of v_c .

The analyses in [9] and in recent theoretical publications (see e.g. [10,11] and references therein) employ the Feynman hypothesis and provide a qualitative explanation of the MIT experimental result. In this Letter we point out a simple geometrical effect which is characteristic for elongated cylindrical traps. Due to the non-uniform character of the radial density profile, the spectrum of axially propagating excitations in these traps is very different from the Bogolyubov dispersion law (2), and this difference leads to a strong decrease of the critical velocity. We show that this effect can at least partially

explain the small critical velocity measured in the MIT experiment [9].

We consider an infinitely long cylindrical condensate which is harmonically trapped in the radial (ρ) direction. Then the condensate wave function $\psi_0(\rho)$ satisfies the Gross-Pitaevskii equation

$$\left(-\frac{\Delta_\rho}{2} + V(\rho) - \mu + g|\psi_0(\rho)|^2 \right) \psi_0(\rho) = 0,$$

where μ is the chemical potential, $V(\rho) = \omega^2 \rho^2 / 2$ is the trapping potential, and ω the trap frequency. In the Thomas-Fermi regime, where the ratio $\eta = \mu/\omega \gg 1$, the density profile is given by $n_0(\rho) \equiv |\psi_0|^2 = (\mu - V(\rho))/g$ and the chemical potential is related to the maximum condensate density as $\mu = n_{0\max}g$.

Elementary excitations can be regarded as quantized fluctuations of the condensate wavefunction [1]. In our trapping geometry they are characterized by the axial (z) wavevector k and radial angular momentum m . The corresponding part of the field operator reads

$$\delta\hat{\psi} = \sum_{m,k} (u_{mk}\hat{b}_{mk} - v_{mk}^*\hat{b}_{mk}^\dagger),$$

where $\hat{b}_{mk}(\hat{b}_{mk}^\dagger)$ are annihilation(creation) operators of the excitations. The excitation wave functions can be written in the form

$$(u, v)_{mk}(\rho, z) = (u, v)_{mk}(\rho) \exp(im\phi) \exp(ikz),$$

where ϕ is the angle in the x, y plane, and the radial functions $u_{mk}(\rho)$ and $v_{mk}(\rho)$ are solutions of the Bogolyubov-de Gennes equations (see e.g. [12])

$$\epsilon u = \left(-\frac{\Delta_\rho}{2} + \frac{k^2}{2} + V - \mu + 2n_0g \right) u + n_0gv, \quad (3)$$

$$-\epsilon v = \left(-\frac{\Delta_\rho}{2} + \frac{k^2}{2} + V - \mu + 2n_0g \right) v + n_0gu. \quad (4)$$

Equations (3) and (4) constitute an eigenvalue problem. For given m and k , they lead to a set of frequencies $\epsilon_{nm}(k)$ characterized by the radial quantum number n which takes integer values from zero to infinity. In the limit $\epsilon_{nm}(k) \ll \mu$, these modes were found for $m = 0$ in the hydrodynamic approach in [13–15]. For $kR \ll 1$, where $R = (2\mu/\omega^2)^{1/2}$ is the Thomas-Fermi radial size of

the condensate, the dispersion relation can be expanded in powers of k^2 [13,15]:

$$\epsilon_{n0}^2(k) = 2\omega^2 n(n+1) + \frac{\omega^2}{4}(kR)^2 + O(k^4). \quad (5)$$

The lowest mode ($n = 0$) represents axially propagating phonons: For this mode we have $\epsilon_{00}(k) = c_Z k$, where the sound velocity $c_Z = \sqrt{n_{0max}g/2}$ is smaller by a factor of $\sqrt{2}$ than the Bogolyubov speed of sound at maximum condensate density, c_s [13–15]. The velocity $c_s/\sqrt{2}$ of axially propagating phonons has been measured in the MIT experiment [16].

The first correction to the linear behavior of the dispersion law $\epsilon_{00}(k)$ at $kR \ll 1$ reveals its negative curvature: $\delta\epsilon_{00} = -\omega(kR)^3/192$ [15]. In [13] the perturbative analysis leading to Eq.(5) was extended numerically to $k \sim 1/R$, still assuming that $\epsilon_{n0}(k) \ll \mu$. The calculations show that the group velocity of the first mode, $d\epsilon_{00}/dk$, monotonously decreases with increasing k and can become significantly smaller than c_Z characteristic for $k \ll 1/R$. This indicates that the critical velocity (1) associated with creating axial excitations ($m = 0, n = 0$) is smaller than c_s and can also be reduced to below c_Z . The physical reason is that the decrease of the condensate density with increasing ρ makes the axial superfluid flow less stable (see below).

The hydrodynamic approach used in [13–15] is not valid for $\epsilon \gtrsim \mu$, where the excitation spectrum is no longer phonon-like and is dominated by the single particle dispersion relation $\epsilon(k) = k^2/2$ (see Eq.(2)). The crossover between the two regimes occurs at $k \sim k_c = \mu^{1/2}$ and prevents the decrease of the group velocity with further increase in k . Obviously, the decrease of the critical velocity due to the radial inhomogeneity of the density profile can be dramatic only for $k_c \gg 1/R$, i.e. in large condensates with $\eta \gg 1$.

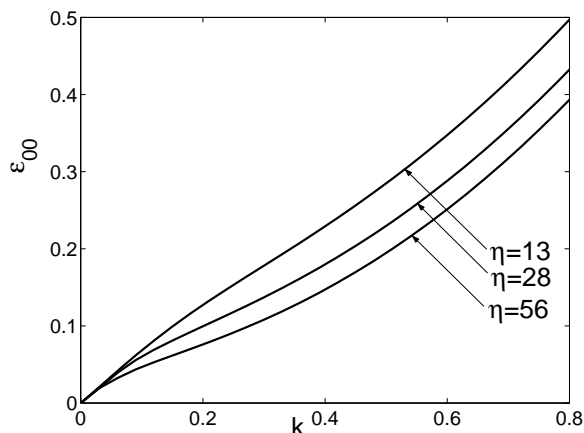


FIG. 1. The dispersion law for the lowest mode ($n = 0, m = 0$) at various values of η . The excitation energy ϵ_{00} is given in units of μ , and the momentum k in units of k_c .

The Bogolyubov-de Gennes equations (3) and (4) are

valid for arbitrary k and hence allow us to find the excitation spectrum in the crossover regime (see Fig.1) and establish the value of the critical velocity as a function of the Thomas-Fermi parameter η . Since the spectrum of excitations consists of a number of independent branches characterized by the quantum numbers n and m , Eq.(1) gives a value $v_c^{(nm)}$ for the critical velocity corresponding to each mode. The results of our numerical calculations for the two lowest modes ($n = 0, m = 0$ and $n = 1, m = 0$) are presented in Fig.2. The breakdown of superfluidity occurs when the velocity of the flow matches the lowest of the velocities $v_c^{(nm)}$, which is proved to be v_c^{00} . The corresponding curve in Fig.2 indicates a significant decrease of the critical velocity compared to c_Z already at $\eta \sim 10$.

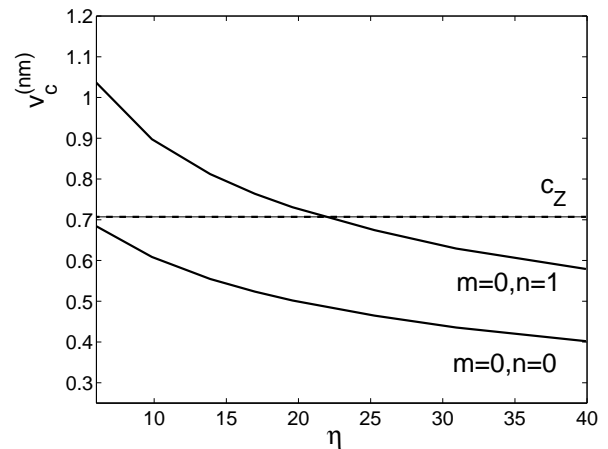


FIG. 2. The critical velocity $v_c^{(nm)}$ (in units of c_s) for the two lowest modes versus the Thomas-Fermi parameter $\eta = \mu/\omega$. The dashed line shows the value of c_Z .

The decrease of the critical velocity with increasing ratio μ/ω seems counterintuitive. One can increase this ratio by decreasing ω and keeping constant μ . Then, at a constant density on the axis of the cylinder, for radially larger condensates one gets smaller v_c . However, this phenomenon has a clear physical explanation. The key point is the non-uniform character of the radial density profile. With increasing axial wavevector k , the wave functions of the excitations with $k \lesssim k_c$ are more localized in the outer spatial region of the condensate and are thus more sensitive to the small value of n_0 in this region. It is this feature that provides a decrease of v_c with increasing η , since for larger η one has more possibilities to increase k and still satisfy the condition $k \lesssim k_c$. The described situation is not met in liquid helium where the density is practically constant and changes only in the region very close to the border of the sample.

We now turn to the discussion of the MIT experiment [9]. The radial frequency in this experiment was $\omega = 2\pi \times 65 \text{ s}^{-1}$ and the chemical potential $\mu = 110 \text{ nK}$, so that $\eta \approx 35$. The sound velocity $c_s = 6.2 \text{ mm/s}$ was

measured by observing a ballistic expansion. For these parameters we find $v_c \approx 0.42c_s = 2.6$ mm/s, which is not far from the observed value $v_c^{(exp)} \approx 1.6$ mm/s [9]. We do not pretend to explain the MIT data as the condensate in the experiment is not an infinite cylinder. The ratio of the radial to axial frequency for the MIT trap is 3.3, which indicates that the effect of a finite axial size can be important. Nevertheless, our calculations may partially explain the small critical velocity observed in the experiment. Moreover, a smaller value of v_c originating from our geometrical effect can facilitate the nucleation of vortex objects and thus provide a larger decrease of the critical velocity.

We would like to thank M. Lewenstein, A. Sanpera and A. Muryshev for fruitful discussions and numerical insights. The work was supported by the Austrian Science Foundation, by the Stichting voor Fundamenteel Onderzoek der Materie (FOM), by INTAS, and by the Russian Foundation for Basic Studies (grant 99-02-1802).

-
- [1] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics, Part 2* (Pergamon Press, Oxford, 1980).
 - [2] R. P. Feynman, *Statistical Mechanics: a set of lectures* (Benjamin Reading, Massachusetts, 1982).
 - [3] R. J. Donnelly, *Quantized vortices in helium II* (Cambridge University Press, Cambridge, 1991).
 - [4] M. H. Anderson *et al.*, Science **269**, 198 (1995).
 - [5] C. C. Bradley, C. A. Sackett, J. J. Tolett, and R. G. Hulet, Phys. Rev. Lett. **75**, 1687 (1995).
 - [6] K. B. Davis *et al.*, Phys. Rev. Lett. **75**, 3969 (1995).
 - [7] W. Ketterle, D. S. Durfee, and D. M. Stamper-Kurn, *Proceedings of the International School of Physics "Enrico Fermi"*, edited by M. Inguscio, S. Stringari, and C. Wieman (ISO press, Amsterdam, 1984).
 - [8] N. Bogolyubov, J. Phys. USSR **11**, 23 (1947).
 - [9] C. Raman *et al.*, e-print cond-mat/9909109.
 - [10] M. Crescimanno, C. G. Koay, R. Peterson, and R. Walsworth, e-print cond-mat/0001163.
 - [11] T. Winiecki, J. F. McCann, and C. Adams, Phys. Rev. Lett. **82**, 5186 (1999).
 - [12] P. Öhberg *et al.*, Phys. Rev. A **56**, R3346 (1997).
 - [13] E. Zaremba, Phys. Rev. A **57**, 518 (1998).
 - [14] G. Kavoulakis and C. Pethick, Phys. Rev. A **58**, 1563 (1998).
 - [15] S. Stringari, Phys. Rev. A **58**, 2385 (1998).
 - [16] M. Andrews *et al.*, Phys. Rev. Lett. **79**, 553 (1997).